Linear Programming by Delayed Column Generation for Bounds on Reliability of Larger Systems

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ABSTRACT: In various efforts to secure the resilience of community, accurate reliability analysis of civil systems is critical considering their pivotal functions. As such systems generally consist of multiple components, their reliability analysis requires complete information to construct joint probabilistic distributions of component events, which is rarely available in practice. In order to obtain the best estimates on the system reliability based on the available information, the linear programming (LP) bounds method was proposed (Song and Der Kiureghian 2003).The method obtains bounds on system reliability by solving LP problems constructed by decomposing the event space into mutually exclusive and collectively exhaustive (MECE) events. Despite the optimality and flexibility of the LP bounds method, there is a limitation in the size of systems as the number of MECE events increases exponentially in regards to that of component events. In order to address this issue, this paper develops an alternative LP bounds formulation by employing *delayed column generation*, in which the LP is solved as an iteration of smaller binary integer programming (BIP). The BIP can be formulated by Boolean algebra that represents the inclusion relationships between component events, system event, and constraint events. The proposed formulation requires polynomial memory in regards to the number of constraints, allowing the evaluation of the LP bounds for larger systems and changing the major bottleneck from the number of components to that of constraint events incorporated into the LP. Four numerical examples are provided to illustrate and demonstrate the proposed method.

1. INTRODUCTION

Civil systems, e.g. structural systems, power networks, and water distribution networks, serve a pivotal role in a community, which highlights the importance of their accurate reliability analysis. As systems consist of multiple components, their reliability analysis requires joint probability distributions of component events. In reality, however, the complete information to formulate high-order joint distributions is rarely available while the information at hand is usually low-order probabilistic information such as uni- and bi-component probabilities. In order to compensate such limitation, it is common to introduce further assumptions to formulate the joint distributions based on given low-order information. There is a risk, however, that such assumptions may not correctly capture the true nature of given systems, leading to inaccurate inference results. In order to eliminate the need for additional assumptions, theoretical bounding methods instead evaluate the bounds on system reliability utilizing only the given information, rather than a specific value (Ditlevsen 1979). However, most of these methods are limited in their applications for they can handle only a limited class of events, e.g. union and intersection events.

On the other hand, the linear programming (LP) bounds method decomposes the event space into mutually exclusive and collectively exhaustive (MECE) events, and evaluates the bounds on system reliability by solving an LP problem (Hailperin 1965; Song and Der Kiureghian 2003). The method is applicable to any definitions of events as the probability of an event can always be expressed as a linear sum of those of MECE events. Furthermore, it has been proved that the LP bounds are the narrowest bounds that can be elicited from any given probabilistic information. However, as the number of MECE events exponentially increases in regards to that of component events, there is a limitation in the size of systems that the method can be applied. It has been reported that systems with up to 17 components can be handled with conventional computers while the method has been applied to numerical examples with up to 12 components in the literature (Song and Der Kiureghian 2005).

In order to evaluate the LP bounds for larger systems, this paper employs the *delayed column generation* (DCG), which is an LP technique developed for problems with a large number of *decision* variables to be optimized (Gilmore and Gomory 1961). Thereby, the LP bounds method is formulated as an iteration of smaller binary integer programming (BIP), utilizing the inclusion relationships between component events, system event, and constraint events. The new optimization problem of BIP demands the polynomial memory in regards to the number of constraints, extending the size of systems for which the LP bounds can be computed. Four numerical examples are provided to illustrate and demonstrate the proposed method.

2. BOUNDS ON SYSTEM RELIABILITY BY LINEAR PROGRAMMING (LP)

Consider a system event E^s and component events E_k having l_k states, $k = 1, \cdots, N$, which together constitute the total event space Ω. When the probabilities of constraint events E_i^c are given as b_i, i.e. $P(E_i^c) = b_i$, $i = 1, \cdots, m-1$, the LP bounds method evaluates the bounds on the probability of system event E^s, $P(E^s)$ by decomposing Ω into MECE events e_j, $j = 1, \cdots, n$, where $n = \prod_{k=1}^{N} l_k$ (Hailperin 1965; Song and Der Kiureghian 2003). Such decomposition allows us to minimize $P(E^s)$ by solving the following LP:

$$\textbf{Problem 1:} \min_{\mathbf{p}} \ \mathbf{c}^{\mathrm{T}}\mathbf{p}$$
$$\text{subject to } \mathbf{A}\mathbf{p} = \mathbf{b}$$
$$\mathbf{p} \in \mathbb{R}_+^n$$

where **p** is the column vector with dimension n, representing the probabilities of the MECE events e_j. The elements of **p** are the decision variables to be optimized. The elements of the column vector **c** with dimension n are determined from the inclusion relationships between e_j and E^s as

$$c_j = \begin{cases} 1, & \text{if } e_j \subseteq E^s \\ 0, & \text{otherwise} \end{cases}, \ j = 1, \cdots, n \qquad (1)$$

On the other hand, j-th column $\mathbf{A}_j$ of $m \times n$ matrix **A**, stands for each MECE event e_j, and the i-th element a_{ij} of $\mathbf{A}_j$, is decided by the relationship between e_j and E_i^c as

$$a_{ij} = \begin{cases} 1, & \text{if } e_j \subseteq E_i^c \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, \cdots, m-1 \text{ and } j = 1, \cdots, n \qquad (2)$$

The last constraint event E_m^c is set as the event space Ω to account for the axiom

$$P(\Omega) = 1 \qquad (3)$$

and since $\forall e_j \subseteq \Omega$, all elements of the last row of **A** is 1, i.e.

$$a_{mj} = 1, j = 1, \cdots, n \qquad (4)$$

The column vector **b** consists of the given probabilities of constraint events, b_i, with its last m-th element being 1. For the moment, inequality constraints and maximizing problem are not considered for concise illustrations. However, with simple modifications, they can also be taken into account as illustrated in Sections 4.1 and 4.2.

It is noted that the dimensions of matrices **c**, **p**, and **A** in **Problem 1**, exponentially increase in regards to N which make the LP bounds method unable to handle systems with a large number of components. In order to address this issue, in Section 3, an alternative formulation of the LP bounds is proposed by employing DCG (Gilmore and Gomory 1961).

3. LP BOUNDS BY DELAYED COLUMN GENERATION (DCG)

In order to deal with a large number of decision variables, DCG avoids explicit enumeration of the matrices **c**, **p**, and **A**, by introducing another optimization problem. First, let us review the concept of *basic feasible solution* (Bertsimas and Tsitsiklis 1997) summarized as follows:

Definition 1 (*Basic feasible solution*): Consider the constraints $\mathbf{Ap}=\mathbf{b}$ for $\mathbf{p}\geq\mathbf{0}$ in **Problem 1**. Given that the rows of **A** are linearly independent, a vector **p** is a basic feasible solution if there exist indices $B(1),\cdots,B(m)$ such that

(1) The columns $\mathbf{A}_{B(1)},\cdots,\mathbf{A}_{B(m)}$ are linearly independent;
(2) $p_j=0,\forall j\neq B(1),\cdots,B(m)$; and
(3) $\mathbf{p}_B=\mathbf{B}^{-1}\mathbf{b}\geq\mathbf{0}$

where the basis matrix **B** is the sub-matrix of **A** consisting of columns $\mathbf{A}_{B(1)},\cdots,\mathbf{A}_{B(m)}$; p_j is j-th element of **p**; and $\mathbf{p}_B$ is the sub-vector of **p** with $p_{B(1)},\cdots,p_{B(m)}$. In the following illustrations, it is assumed that the constraints of the optimization problem are given in a way that the rows of **A** are linearly independent, and there exists an optimal feasible solution (with finite values) for the given problem.

Given a basic feasible solution **p**, the *reduced cost* $\bar{c}_j$ of p_j is defined as

$$\bar{c}_j=c_j-\bar{\mathbf{y}}\mathbf{A}_j,\ j=1,\cdots,n \tag{5}$$

where $\bar{\mathbf{y}}=\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$ and the basis cost vector $\boldsymbol{c}_B$ is the sub-vector of **c** that corresponds to the basic variables. Then, the optimality of **p** can be checked as follows (Bertsimas and Tsitsiklis 1997):

Theorem 1 (*Optimality of basic feasible solution*): Consider the basis matrix **B**, and the basis cost vector $\boldsymbol{c}_B$ for a basic feasible solution **p**. Then, **p** is optimal if $\bar{c}_j\geq 0$ for all $j=1,\cdots,n$.

Theorem 1 suggests that the matrices **B**, $\boldsymbol{c}_B$, and $\mathbf{p}_B$, whose dimensions are polynomial in regards to the number of constraints, *m*, are sufficient to specify the current solution and check its optimality. When it is concluded that the minimum value of $\bar{c}_j$ is non-negative, the optimization is terminated as the current **p** is optimal. Otherwise, if an index j^* is found such that $\bar{c}_{j^*}<0$, the k^*-th basis solution is replaced by j^*, being determined by

$$k^*=\arg\min_{\{k=1,\cdots,m|u_k>0\}}\frac{p_{B(k)}}{u_k} \tag{6}$$

where u_k is k-th element of vector $\mathbf{u}=\mathbf{B}^{-1}\mathbf{A}_{j^*}$. In other words, $\mathbf{A}_{B(k^*)}$ in **B** and $c_{B(k^*)}$ in $\mathbf{c}_B$ are replaced by $\mathbf{A}_{B(j^*)}$ and $c_{B(j^*)}$, respectively. After updating the basis solution, the optimization is continued until the optimality condition in **Theorem 1** is achieved.

3.1. Representation of the set of columns in constraint matrix by polyhedra

By employing Boolean algebra (Brown 2003), $\mathcal{A}_1$, the set of columns in **A**, and $\mathcal{A}_2$, the set of those associated with a fixed cost of either 0 or 1, can be specified by polyhedra, i.e. the set of linear inequalities, as $\mathcal{A}_h=\{\mathbf{a}\in\mathbb{B}^m|\mathbf{A}_h\mathbf{a}\leq\mathbf{b}_h\}$, $h=1,2$. Such representation allows the formulation of an efficient optimization problem to identify the column with minimum reduced cost, which is illustrated in Section 3.2. To this end, the first *N* constraint events are set as the *N* component events, i.e. $E_k^c=E_k$, for $k=1,\cdots,N$. When there is no constraint for *k*-th component event, a dummy constraint can be added as

$$P(E_k^c)\leq 1 \tag{7}$$

It is noted that *i*-th element of **a**, a_i, $i=1,\cdots,m$, is associated with *i*-th constraint event E_i^c, indicating whether the MECE event represented by **a**, is a subset of E_i^c. By the specification of the first *N* constraint events, the values of a_i, $i=N+1,\cdots,m$, can be determined by those of the first *N* elements a_k, $k=1,\cdots,N$.

3.1.1. The universal set of columns

$\mathcal{A}_1$ can be formulated based on the inclusion relationships between constraint events and component events. Consider the constraint event of union, i.e.

$$E_i^c=\bigcup_{k\in\boldsymbol{X}_i}E_k,\ \boldsymbol{X}_i\subseteq\{1,\cdots,N\} \tag{8}$$

Then, the events have the relationships

$$E_k \subseteq E_i^c \text{ for } k \in \boldsymbol{X}_i\text{; and} \\ E_i^c \subseteq \bigcup_{k\in\boldsymbol{X}_i} E_k \tag{9}$$

Boolean algebra can explain these relationships by a polyhedron of the elements in **a**, i.e.

$$-a_i + a_k \le 0 \text{ for } k \in \boldsymbol{X}_i\text{; and} \\ a_i - \sum_{k\in\boldsymbol{X}_i} a_k \le 0 \tag{10}$$

On the other hand, the constraint event of intersection, i.e.

$$E_i^c = \bigcap_{k\in\boldsymbol{X}_i} E_k, \boldsymbol{X}_i \subseteq \{1,\cdots,N\} \tag{11}$$

has the relationships with component events such that

$$E_i^c \subseteq E_k \text{ for } k \in \boldsymbol{X}_i\text{; and} \\ \bigcap_{k\in\boldsymbol{X}_i} E_k \subseteq E_i^c \tag{12}$$

This leads to the linear inequalities of

$$a_i - a_k \le 0 \text{ for } k \in \boldsymbol{X}_i\text{; and} \\ -a_i + \sum_{k\in\boldsymbol{X}_i} a_k \le |\boldsymbol{X}_i| - 1 \tag{13}$$

Consider more general constraint events determined in terms of cut-sets, which are the unions of intersection events, i.e.

$$E_i^c = \bigcup_{r=1}^{N_{\text{sub}}} \left(\bigcap_{k\in\boldsymbol{X}_i^r} E_k \right), \\ \boldsymbol{X}_i^r \subseteq \{1,\cdots,N\} \tag{14}$$

where N_{sub} is the number of sub-events. Then, for each intersection event, a dummy constraint event in Eq. (7), can be introduced, and the polyhedron can be formulated from the relationships between the component events and those artificial events, and subsequently the artificial ones and the cut-set event. This scheme is equally applicable to those of link-sets, which is the counterpart of cut-sets, i.e.

$$E_i^c = \bigcap_{r=1}^{N_{\text{sub}}} \left(\bigcup_{k\in\boldsymbol{X}_i^r} E_k \right), \\ \boldsymbol{X}_i^r \subseteq \{1,\cdots,N\} \tag{15}$$

In this case, the dummy constraints are introduced for each union event.

Lastly, the element a_m associated with Ω, should always be 1 from Eq. (4), leading to the inequalities

$$a_m \le 1 \text{ and } -a_m \le -1 \tag{16}$$

For the constraint events E_i^c, $i = N+1,\cdots,m$, a set of inequalities derived by Eqs. (10), (13), and (16), can be collected as the rows of $\mathbf{A}_1$ and $\mathbf{b}_1$.

3.1.2. The subset of columns with identical cost

In the following optimization problem in Section 3.2, the set of columns, $\mathcal{A}_2$, all of which have the identical cost c_j, i.e. either 0 or 1, is required. The inequalities to sort out such columns can be derived from the relationships between component events and system event. The matrices $\mathbf{A}_2$ and $\mathbf{b}_2$ can be constructed for $\mathcal{A}_2$, by adding the new inequalities that account for such relationships, to $\mathbf{A}_1$ and $\mathbf{b}_1$.

When a system event E^s is a union event in the form of Eq. (8) with the index set $\boldsymbol{X}^s$, the system event is false, i.e. $c_j = 0$, if all associated component events are false. This condition can be expressed as

$$\sum_{k\in\boldsymbol{X}^s} a_k \le 0 \text{ for } c_j = 0 \tag{17}$$

On the other hand, the system event is true when at least one event is true, i.e.

$$-\sum_{k\in\boldsymbol{X}^s} a_k \le -1 \text{ for } c_j = 1 \tag{18}$$

The reduced set for a system event of intersection as in Eq. (11) with the index set $\boldsymbol{X}^s$, can be formulated in the same way as

$$\sum_{k\in\boldsymbol{X}^s} a_k \le |\boldsymbol{X}^s| - 1 \text{ for } c_j = 0\text{; and} \\ -\sum_{k\in\boldsymbol{X}^s} a_k \le -|\boldsymbol{X}^s| \text{ for } c_j = 1 \tag{19}$$

In the case of system events of cut- and link-sets, dummy constraint events of Eq. (7) can be utilized as discussed for constraint events in Section 3.1.1.

Finally, for *k-out-of-N:G* systems which are false when less than k component events are true, the columns can be sorted out by the inequalities

$$\sum_{k\in\boldsymbol{X}^s} a_k \le k - 1 \text{ for } c_j = 0\text{; and} \\ -\sum_{k\in\boldsymbol{X}^s} a_k \le -k \text{ for } c_j = 1 \tag{20}$$

The modification of Eq. (20) for *k-out-of-N:F* systems, i.e. the system event is true if less than k component events are false, is straightforward.

3.2. LP as an iteration of binary integer programming (BIP)

Integer programming (IP) is a class of optimization problem in which the decision variables are integer, while the problem is formulated by a linear objective function and a polyhedron as the set of feasible solutions. BIP is a subset of IP whose decision variables are further constrained as binary. Utilizing the polyhedra of column sets formulated in Section 3.1, and ignoring the first term c_j in Eq. (5), the minimizing problem of reduced cost can be formulated as BIP:

Problem 2: $\bar{\bar{\mathbf{c}}}^* = \max_{\mathbf{a}} \bar{\mathbf{y}}\mathbf{a}$
subject to $\boldsymbol{A}_h\boldsymbol{a} \le \mathbf{b}_h$
$\mathbf{a} \in \mathbb{B}^m$

where h indicates the index of column set over which the optimization is performed. This new optimization problem leads to the following algorithm to evaluate the LP bounds by DCG:

Algorithm 1 (*LP bounds by iteration of BIPs – minimizing problem*):
Given initial basis matrix $\mathbf{B}$, $\mathcal{A}_h = \{\mathbf{a} \in \mathbb{B}^m | \mathbf{A}_h\mathbf{a} \le \mathbf{b}_h\}$ for $h = 1,2$,
1. Evaluate $\mathbf{c}_B$ and $\bar{\mathbf{y}} = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$ for current $\mathbf{B}$.
2. Optimize **Problem 2** for $\mathcal{A}_1$ to get $\bar{\bar{\mathbf{c}}}^*$; and the associated column $\mathbf{a}^*$, cost c^*, and reduced cost $\bar{c}^*$.
   2-1. If $\bar{c}^* < 0$, go to Step 4;
   2-2. else if $0 < \bar{c}^* \le 1$ and $c^* = 1$, go to Step 3;
   2-3. else, current $\mathbf{B}$ is optimal and terminate the iteration.
3. Optimize **Problem 2** for $\mathcal{A}_2$ with cost 0, and re-evaluate $\bar{\bar{\mathbf{c}}}^*$, $\mathbf{a}^*$, c^*, and $\bar{c}^*$.
   3-1. If $\bar{c}^* < 0$, go to Step 4;
   3-2. else, the current $\mathbf{B}$ is optimal and terminate the iteration.
4. Evaluate $\mathbf{u} = \mathbf{B}^{-1}\mathbf{a}^*$, and decide the index k^* by Eq. (6). Replace the column $\mathbf{A}_{B(k^*)}$ in $\mathbf{B}$ by $\mathbf{a}^*$. Go back to Step 1.

BIP is a well-explored class of optimization problems for which various general-purpose software programs are available. In the following numerical examples, the academic version of cplex® 12.8.0 was used to solve **Problem 2**.

4. ADDITIONAL REMARKS

4.1. Inequality constraint events

Consider the constraint events E_i^c for $i \in \boldsymbol{Y}_1 \subseteq \{1, \cdots, m-1\}$ for which the inequality constraints are given as

$$P(E_i^c) \le b_i,\ i \in \boldsymbol{Y}_1 \subseteq \{1, \cdots, m-1\} \quad (21)$$

They can be expressed as equalities by introducing slack variables $s_i \ge 0$, i.e.

$$P(E_i^c) + s_i = b_i,\ i \in \boldsymbol{Y}_1 \quad (22)$$

Therefore, the inequality constraints of Eq. (21), can be considered in the proposed method, by introducing an additional column set $\mathcal{A}_3$ for the slack variables, i.e.

$$\mathcal{A}_3 = \bigcup_{i \in \boldsymbol{Y}_1}\{\mathbf{e}_i\} \quad (23)$$

where $\mathbf{e}_i$ is the unit vector of dimension m with its i-th element 1 while others are zero. On the other hand, for the inequality constraints in the form of

$$P(E_i^c) \ge b_i,\ i \in \boldsymbol{Y}_2 \subseteq \{1, \cdots, m-1\} \quad (24)$$

the slack variables $s_i \ge 0$ are introduced as

$$P(E_i^c) - s_i = b_i,\ i \in \boldsymbol{Y}_2 \quad (25)$$

In this case, the column set $\mathcal{A}_4$ is defined as

$$\mathcal{A}_4 = \bigcup_{i \in \boldsymbol{Y}_2}\{-\mathbf{e}_i\} \quad (26)$$

It is noted that all columns in $\mathcal{A}_3$ and $\mathcal{A}_4$ have cost $c = 0$ as they do not represent MECE events, and **Problem 2** for $\mathcal{A}_3$ ($\mathcal{A}_4$), can be evaluated by inspection over the i-th elements of $\bar{\mathbf{y}}$, $\bar{y}_i$ for $i \in \boldsymbol{Y}_1$ ($i \in \boldsymbol{Y}_2$). Consequently, Step 2 of **Algorithm 1** needs to be executed for these two sets along with $\mathcal{A}_1$. When multiple columns with negative reduced costs are found, it is customary to update the basic solution by the one with the smallest reduced cost.

4.2. Maximizing problem

For maximization, the reduced cost is defined as

$$\bar{c}_j = -c_j + \bar{\mathbf{y}}\mathbf{A}_j \quad (27)$$

which has the opposite sign with Eq. (5), and leads to **Problem 3** by modifying **Problem 2**:

Problem 3: $\bar{\bar{\mathbf{c}}}^* = \min_{\mathbf{a}} \bar{\mathbf{y}}\mathbf{a}$
subject to $\boldsymbol{A}_h\boldsymbol{a} \le \mathbf{b}_h$
$\mathbf{a} \in \mathbb{B}^m$

Accordingly, Steps 2, 2-2, and 3 of **Algorithm 1**, need to be modified as follows:

2. Optimize **Problem 3** for $\mathcal{A}_1$.
2-2. else if $0 \le \bar{c}^* < 1$ and $c^* = 0$, go to Step 3.
3. Optimize **Problem 3** for $\mathcal{A}_2$ with cost 1.

4.3. Finding initial basis matrix

Execution of **Algorithm 1** requires the initial basis matrix **B**. However, it is often not straightforward to find a feasible solution for a large LP. This issue can be addressed by introducing artificial variables that form the column set $\mathcal{A}_5$ as

$$\mathcal{A}_5 = \{\mathbf{a} \in \mathbb{B}^m \mid |\mathbf{a}| = 1\} \cap (\{\mathbf{e}_m\} \cup \mathcal{A}_3)^c \quad (28)$$

i.e. the set of unit vectors that are not the elements of $\mathcal{A}_1$ and $\mathcal{A}_3$. This new set is incorporated in Step 2 of **Algorithm 1** with all of its elements attributed to cost 1 while those of other sets $\mathcal{A}_h$, $h = 1, \cdots, 4$, all having cost 0. Then, the algorithm starts with $m \times m$ identity matrix as **B**. If the given problem is feasible and bounded, the algorithm is terminated with **B** that does not include the columns in $\mathcal{A}_5$. **Algorithm 1** can then be executed to evaluate the LP bounds with this basis matrix.

5. NUMERICAL EXAMPLES

In the following examples except the first one, uni- and bi-component probabilities are utilized as constraint events, all in equality. To compute the Kounias-Hunter-Ditlevsen (KHD) bounds in Sections 5.2 and 5.3 for comparison (Ditlevsen 1979), the optimal order of component events has been determined by greedy search starting at each component event as the width of KHD bounds depends on the ordering of components.

5.1. Illustrative example: a series system with three component events

For illustrative purpose, consider the evaluation of lower bound on the probability of a series system event E^s consisting of three component events, i.e. $E^s = E_1 \cup E_2 \cup E_3$. The information given for this system is

$$P(E_1) = 0.5,\ P(E_2) = 0.2, \quad (29)$$
$$P(E_3) = 0.4, \text{ and } P(E_1 \cap E_2) = 0.1$$

which sequentially correspond to the constraint events E_i^c for $i = 1, \cdots, 4$, with the last constraint event $E_5^c = \Omega$. From Eqs. (13) and (16), the polyhedron for $\mathcal{A}_1$ is specified by the two matrices $\mathbf{A}_1$ and $\mathbf{b}_1$ as

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } \mathbf{b}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad (30)$$

The column set $\mathbf{A}_2$ with cost 0, can be derived by adding rows to $\mathbf{A}_1$ and $\mathbf{b}_1$ following Eq. (17) as

$$\mathbf{A}_2 = \begin{bmatrix} & & \mathbf{A}_1 & & \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{b}_2 = \begin{bmatrix} \mathbf{b}_1 \\ 0 \end{bmatrix} \quad (31)$$

Suppose that the initial basis matrix **B** is obtained either by observation or by the strategy illustrated in Section 4.3, as

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad (32)$$

In regards to **B**, the vectors $\mathbf{c}_B$, $\mathbf{p}_B$, and $\bar{\mathbf{y}}$ are evaluated as

$$\begin{gathered} \mathbf{c}_B = [1 \quad 1 \quad 1 \quad 1 \quad 0]^{\mathrm{T}}, \\ \mathbf{p}_B = [0.4 \quad 0.1 \quad 0.3 \quad 0.1 \quad 0.1]^{\mathrm{T}}, \text{ and} \\ \bar{\mathbf{y}} = [1 \quad 1 \quad 1 \quad -2 \quad 0] \end{gathered} \quad (33)$$

where the objective value $P(E^s) = \mathbf{c}_B^{\mathrm{T}} \mathbf{p}_B = 0.9$. Then, the execution of Step 2 in **Algorithm 1** gives the result:

$$\begin{gathered} \mathbf{a}^* = [1 \quad 0 \quad 1 \quad 0 \quad 1]^{\mathrm{T}} \\ \text{with } \bar{\bar{\mathbf{c}}}^* = 2,\ c^* = 1, \text{ and } \bar{c}^* = -1 \end{gathered} \quad (34)$$

For $\bar{c}^* < 0$, $\mathbf{a}^*$ replaces the third column of **B** following Eq. (6). After another iteration of **Algorithm 1** with the new basis matrix, one can conclude that there is no column with negative reduced cost, i.e. the current solution is optimal. For the new basis matrix, the vectors are evaluated as

$$\begin{gathered} \mathbf{c}_B = [1 \quad 1 \quad 1 \quad 1 \quad 0]^{\mathrm{T}}, \text{ and} \\ \mathbf{p}_B = [0.1 \quad 0.1 \quad 0.3 \quad 0.1 \quad 0.4]^{\mathrm{T}} \end{gathered} \quad (35)$$

with the optimal solution $P(E^s) = 0.6$.

5.2. Truss bridge structure as a series system

Consider the example truss bridge structure consisting of 25 members in Figure 1, where loads are applied on joints 1, 2, and 3. In the system, the random variables (r.v.'s) are: the yield stresses of members, σ_{yk}, $k = 1, \cdots, 25$, and the magnitude of P_1, P_2, and P_3. Their distribution types and statistical parameters are summarized in Table 1. The yield stresses and loads are assumed to be statistically independent while being dependent within each group. The cross section areas of members are assumed as 15×10^{-4} m^2 for members 1, ⋯, 6; 14×10^{-4} m^2 for members 7, ⋯, 12; and 12×10^{-4} m^2 for members 13, ⋯, 17.

The system event is defined as a series system of the component failure events. By applying the proposed method, the LP bounds are evaluated as $[8.566, 9.026] \times 10^{-2}$ whose upper bound is tighter than the KHD bounds of $[8.566, 9.050] \times 10^{-2}$. It is noted that 42 constraint events of bi-component probabilities among 300, along with the 25 uni-component probabilities, are sufficient to obtain the optimal bounds, suggesting that one may not need to incorporate all constraint events to get the solution. Such a selective use of constraint events makes the proposed method applicable to even larger systems.

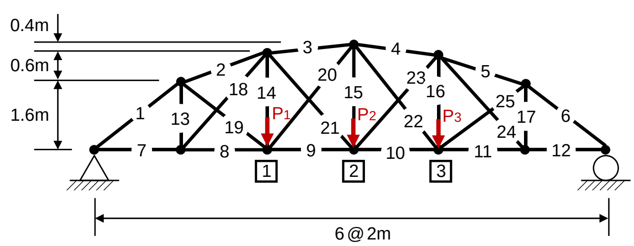

Figure 1. Example truss bridge structure

Table 1. Distribution types and statistical parameters

Random variables	*Distri-bution*	*Mean*	*c.o.v.*	*Correlation coefficient*
P_k, $k = 1,2,3$	*Normal*	*110 kN*	*0.1*	*0.8*
σ_{yk}, $k = 1, \cdots, 25$	*Normal*	*276 MPa*	*0.1*	*0.3*

5.3. Daniels' system

For Daniels' system in Figure 2 which consists of a bundle of N ideally brittle wires, it is assumed that (1) the uncertain wire strengths are statistically independent and identically distributed r.v.'s, and (2) the deterministic load L is equally distributed over the remaining wires (Daniels 1945). The wire strengths are assumed to follow the Weibull distribution with the cumulative distribution function $F(x) = 1 - \exp(-\lambda x^{\beta})$, $x \geq 0$ with parameters $\lambda = 0.01$ and $\beta = 5$. The load L is assumed as 35.

In Figure 3, the results by the LP bounds and the KHD bounds are compared for Daniels' system with different numbers of wires. For the y-axis is in log-scale, the lower bounds of 0 are not presented in the graph. It is noted that the LP bounds provide tighter lower bounds than the KHD bounds. Furthermore, the width of bounds is not insignificant, while the system reliability analytically evaluated (black line) from the above two assumptions, lie within the bounds. This implies that estimating high-order probabilities with inaccurate assumptions, may cause significant errors in inference results.

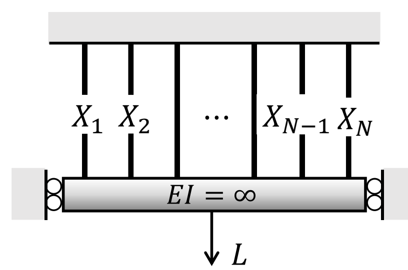

Figure 2. Daniels' system with N wires

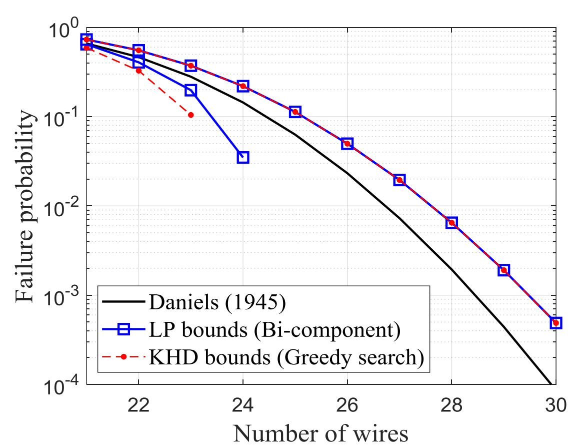

Figure 3. System failure probability evaluated by analytic formulation, LP bounds, and KHD bounds

5.4. k out of N: G system

The LP bounds on the failure probability of a *k-out-of-N:G* systems are obtained where $k = 18$ and $N = 20$. The 20 identical components are assumed to have the failure probability 1×10^{-4} and joint failure probability 0.5×10^{-4}. The LP bounds are evaluated as $[0.4723,\ 4.166] \times 10^{-4}$. On the other hand, based on the correlated binomial distribution model whose parameters are determined by the uni- and bi-component probabilities (Diniz et al. 2010), the system failure probability is calculated as 0.5×10^{-4}, being much closer to the lower bound. Such result highlights the advantage of bounding methods especially in the case where there is not enough information to formulate the joint distributions.

6. CONCLUSIONS

System reliability bounding methods are advantageous in that no additional assumptions are required to formulate the complete joint distributions while utilizing only the given information. In particular, the LP bounds method has been proved by various applications that the method can handle any definition of events, and provides the narrowest bounds possible by given probabilistic information (Song and Der Kiureghian 2003). However, the method has a limitation in terms of the size of systems, since the number of the parameters of the optimization problem, exponentially increases as that of component events increases.

In order to overcome this limitation, the delayed column generation (DCG) technique is employed for the evaluation of LP bounds. The LP is then formulated as an iteration of binary integer programming (BIP) where the BIP is derived from the inclusion relationships between the events of analysis. As a result, the formulation of BIP and the generation of matrices required during the iteration, demand polynomial memory in regards to the number of constraints. This enables us to evaluate LP bounds on larger systems than the original formulation. Four numerical examples are provided to illustrate and demonstrate the proposed method.

As the results of numerical examples suggest, a subset of given constraints can be sufficient to obtain the optimal bounds. This implies that the proposed method is expected to evaluate even larger systems for the number of constraints is now the major bottleneck for the proposed method.

7. ACKNOWLEDGEMENT

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